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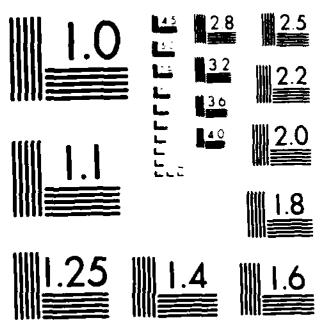
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CENTER FOR STOCHASTIC PROCESSES

Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

ON LIMITING DISTRIBUTIONS OF ORDER STATISTICS WITH VARIABLE RANKS FROM STATIONARY SEQUENCES

by

Shihong Cheng

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ITEM #20, CONTINUED:

Let $\{X_n\}$ be a stationary sequence and $X_1^{(n)} \le \ldots \le X_n^{(n)}$ be the order statistics of $X_1, \ldots, X_n$. In this paper, the limiting distribution of $X_{k_n}^{(n)}$, where $k_n \to \infty$, $k_n/n \to \lambda$, $0 \le \lambda \le 1$ is discussed under distributional mixing conditions. For stationary normal sequences, the limiting distribution of $X_{k_n}^{(n)}$, where $k_n/n \to \lambda \in (0,1)$, is a normal with mean zero and variance

$$\sigma_\lambda^2 = 1 + \frac{1}{\pi\lambda(1-\lambda)} \sum_{n=0}^{\infty} \int_0^{r_n} \frac{\exp\{-a_\lambda^2/(1+r)\}}{(1-r^2)^{1/2}}\, dr$$

if the covariance $\{r_n\}$ converges to zero as fast as $n^{-\rho}$, $\rho>4$, a_λ being the λ-percentile of the standard normal distribution.

ON LIMITING DISTRIBUTIONS OF ORDER STATISTICS WITH VARIABLE RANKS FROM STATIONARY SEQUENCES

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Abstract

Let $\{X_n\}$ be a stationary sequence and $X_1^{(n)} \leq \ldots \leq X_n^{(n)}$ be the order statistics of $X_1, \ldots, X_n$. In this paper, the limiting distribution of $X_{k_n}^{(n)}$, where $k_n \to \infty$, $k_n/n \to \lambda$, $0 \leq \lambda \leq 1$ is discussed under distributional mixing conditions. For stationary normal sequences, the limiting distribution of $X_{k_n}^{(n)}$, where $k_n/n \to \lambda \in (0,1)$, is a normal with mean zero and variance

$$\sigma_\lambda^2 = 1 + \frac{1}{\pi\lambda(1-\lambda)} \sum_{n=0}^{\infty} \int_0^{r_n} \frac{\exp\{-a_\lambda^2/(1+r)\}}{(1-r^2)^{1/2}}\, dr$$

if the covariance $\{r_n\}$ converges to zero as fast as $n^{-\rho}$, $\rho>4$, a_λ being the λ-percentile of the standard normal distribution.

Keywords: Order statistics, stationary sequences, limiting distribution, variable ranks.

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Let $\{X_n\}$ be a sequence of random variables and $X_1^{(n)} \le \ldots \le X_n^{(n)}$ be the order statistics of $X_1, \ldots, X_n$. In this paper it is assumed that the sequence $\{X_n\}$ is stationary and that the ranks k_n of the order statistics $\{X_{k_n}^{(n)}\}$ satisfy the following condition:

$$k_n \to \infty,\ n - k_n \to \infty,\ k_n/n \to \lambda,\ 0 \le \lambda \le 1 .$$

Since the case $\lambda=1$ is easily transformed to the case $\lambda=0$, we discuss only the cases:

$$k_n \to \infty,\ k_n/n \to \lambda,\ 0 \le \lambda < 1 . \tag{0.1}$$

The case $\lambda=0$ has been discussed by Watts, Rootzén and Leadbetter [7]. The case $0<\lambda<1$ has been discussed by the present author [2], but the mixing condition in [2] is hard to check. Here we consider the cases $\lambda=0$ and $0<\lambda<1$ simultaneously, under a distributional mixing condition used by Leadbetter [4].

§1. Notation, assumptions, and introduction

Let $\{X_n\}$ be a stationary sequence with finite dimensional distribution functions, $\{F_{j_1 \ldots j_p}(x_1, \ldots, x_p),\ 1 \le j_1 < j_2 < \ldots\}$, and, in particular, marginal distribution function $F_1(x) = F(x)$. Suppose that $\{u_n\}$ is a real sequence such that

$$\lim_n \frac{n}{\sqrt{k_n}} [F(u_n) - \frac{k_n}{n}] = u\sqrt{1-\lambda} \quad , \quad -\infty < u < \infty . \tag{1.1}$$

The distributional mixing coefficients of $\{X_n\}$ with $\{u_n\}$ are defined by

$\alpha(n,\ell) =$

$$\sup\{|F_{i_1 \ldots i_p j_1 \ldots j_q}(u_n) - F_{i_1 \ldots i_p}(u_n) F_{j_1 \ldots j_q}(u_n)| :\ 1 \le i_1 < \ldots < i_p < j_1 < \ldots < j_q \le n,\ j_1 - i_p > \ell\}$$

where $F_{j_1 \ldots j_p}(u_n) \equiv F_{j_1 \ldots j_p}(u_n, \ldots, u_n)$ for any $1 \le j_1 < j_2 < \ldots < j_p \le n$. Let $\sigma(A_1, \ldots, A_k)$ denote the field generated by sets $A_1, \ldots, A_k$ and

$\beta(n,\ell) =$

$$\sup\{|p(AB) - p(A)p(B)| :\ A \in \sigma(\{X_j \le u_n\}, j=1,\ldots,k),\ B \in \sigma(\{X_j \le u_n\}, j=k+\ell,\ldots,n),\ 1 \le k < k+\ell \le n\}$$

Lemma 1.1 For any measurable sets $A_1, \ldots, A_k$, $B_1, \ldots, B_\ell$ let

$$\alpha = \sup\{|P(A_{i_1}\ldots A_{i_s}B_{j_1}\ldots B_{j_t}) - P(A_{i_1}\ldots A_{i_s})P(B_{j_1}\ldots B_{j_t})|:$$

$$1\le i_1<\ldots<i_s\le k,\ 1\le j_1<\ldots<j_t\le \ell\} .$$

Then we have

$$(1.2)\qquad |P(A_{i_1\ldots i_s}B_{j_1\ldots j_t}) - P(A_{i_1\ldots i_s})P(B_{j_1\ldots j_t})| \le 2^{s+t}\alpha$$

for any s, $1\le s\le k$ and t, $1\le t\le \ell$, where

$$A_{i_1\ldots i_s} = [\bigcap_{i\in\{i_1,\ldots,i_s\}} \bar{A}_i] \cap [\bigcap_{i\bar{\in}\{i_1,\ldots,i_s\}} A_i]$$

$$B_{j_1\ldots j_t} = [\bigcap_{j\in\{j_1,\ldots,j_t\}} \bar{B}_j] \cap [\bigcap_{j\bar{\in}\{j_1,\ldots,j_t\}} B_j] .$$

Proof: It is easy to show that for any sets $S, S_1, \ldots, S_n$,

$$P(S\bar{S}_1\ldots\bar{S}_n) = P(S) - \sum_{p=1}^{n}(-1)^{p-1}\sum_{1\le i_1<\ldots<i_p\le n} P(SS_{i_1}\ldots S_{i_p}) .$$

Hence if $\{i_1,\ldots, i_s\} = \{1,\ldots,S\}$, $\{j_1,\ldots, j_t\} = \{1,\ldots, t\}$, (1.2) is obtained from

$$|P(\bar{A}_1\ldots\bar{A}_sA_{s+1}\ldots A_k\bar{B}_1\ldots\bar{B}_tB_{t+1}\ldots B_\ell) - P(\bar{A}_1\ldots\bar{A}_sA_{s+1}\ldots A_k)P(\bar{B}_1\ldots\bar{B}_tB_{t+1}\ldots B_\ell)|$$

$$\le |P(A_{s+1}\ldots A_kB_{t+1}\ldots B_\ell) - P(A_{s+1}\ldots A_k)P(B_{t+1}\ldots B_\ell)|$$

$$+ \sum_{p=1}^{s}\sum_{1\le i_1<\ldots<i_p\le s}|P(A_{i_1}\ldots A_{i_p}A_{s+1}\ldots A_kB_{t+1}\ldots B_\ell)-P(A_{i_1}\ldots A_{i_p}A_{s+1}\ldots A_k)P(B_{t+1}\ldots B_\ell)|$$

$$+ \sum_{q=1}^{t}\sum_{1\le j_1<\ldots<j_q\le t}|P(A_{s+1}\ldots A_kB_{j_1}\ldots B_{j_q}B_{t+1}\ldots B_\ell)-P(A_{s+1}\ldots A_k)P(B_{j_1}\ldots B_{j_q}B_{t+1}\ldots B_\ell)|$$

$$+ \sum_{p=1}^{s}\sum_{1\le i_1<\ldots<i_p\le s}\sum_{q=1}^{t}\sum_{1\le j_1<\ldots<j_q\le t}|P(A_{i_1}\ldots A_{i_p}A_{s+1}\ldots A_kB_{j_1}\ldots B_{j_q}B_{t+1}\ldots B_\ell)$$

$$- P(A_{i_1}\ldots A_{i_p}A_{s+1}\ldots A_k)P(B_{j_1}\ldots B_{j_q}B_{t+1}\ldots B_\ell)|$$

$$\le \alpha \sum_{p=0}^{s}\binom{s}{p}\sum_{q=0}^{t}\binom{t}{q} \le 2^{s+t}\alpha .$$

Denote the indicator of the set A by I_A and write

$$I_{nj} = I_{\{X_j \le u_n\}} \ , \quad \tilde{I}_{nj} = I_{nj} - F(u_n) \ , \quad \bar{I}_{nj} = \tilde{I}_{nj}/k_n^{1/2} \ , \quad j=1,\ldots,n.$$

Let ℓ_n and $\tilde{\ell}_n$ be two sequences of positive integers such that $\ell_n \le \tilde{\ell}_n \le n$. Define

$$\bar{\xi}_{ni} = \sum_{j=(i-1)(\ell_n+\tilde{\ell}_n)+1}^{(i-1)(\tilde{\ell}_n+\ell_n)+\tilde{\ell}_n} \bar{I}_{nj} \ , \quad \bar{\eta}_{ni} = \sum_{j=(i-1)(\tilde{\ell}_n+\ell_n)+\tilde{\ell}_n+1}^{i(\ell_n+\tilde{\ell}_n)} \bar{I}_{nj} \ , \quad i=1,\ldots,N_n$$

and $\bar{\zeta}_n = \sum_{j=N_n(\ell_n+\tilde{\ell}_n)+1}^{n} I_{nj}$, where $N_n = [\frac{n}{\tilde{\ell}_n+\ell_n}]$. To obtain our results we need to discuss the limiting distributions of $\sum_{i=1}^{N_n} \bar{\xi}_{ni}$, $\sum_{i=1}^{N_n} \bar{\eta}_{ni}$ and $\bar{\zeta}_n$. As preliminaries, we obtain the following lemmas.

Lemma 1.2 The following inequalities hold:

$$(1.3) \quad |Ee^{i\sum_{k=1}^{N_n} \bar{\xi}_{nk} t} - (Ee^{i\bar{\xi}_{n1} t})^{N_n}| \le (n/\tilde{\ell}_n)\cdot\beta(n,\ell_n)$$

$$(1.4) \quad |Ee^{i\sum_{k=1}^{N_n} \bar{\xi}_{nk} t} - (Ee^{i\bar{\xi}_{n1} t})^{N_n}| \le 3^n \ \alpha(n,\ell_n) \ .$$

The above statements are still true if we use η_{nk}, $k=1,\ldots,N_n$ instead of ξ_{nk}, $k=1,\ldots,N_n$ in (1.3) and (1.4).

Proof: By Dvoretzky's lemma 5.3 in [3], it follows that

$$|Ee^{i\sum_{k=1}^{N_n} \bar{\xi}_{nk} t} - (Ee^{i\bar{\xi}_{n1} t})^{N_n}|$$

$$\le \sum_{r=1}^{N_n} |Ee^{i\sum_{k=1}^{r} \bar{\xi}_{nk} t} - (Ee^{i\sum_{k=1}^{r-1} \bar{\xi}_{nk} t})(Ee^{i\bar{\xi}_{n1} t})|$$

$$\le \sum_{r=1}^{N_n} \beta(n,\ell_n) \le (n/\tilde{\ell}_n)\cdot\beta(n,\ell_n) \ .$$

Noticing that

$$|Ee^{i\sum_{k=1}^{N_n}\bar{\eta}_{nk}t} - (Ee^{i\bar{\eta}_{n1}t})^{N_n}| \le (n/\tilde{\ell}_n)\cdot\beta(n,\tilde{\ell}_n) \le (n/\tilde{\ell}_n)\cdot\beta(n,\ell_n) ,$$

we see that (1.3) holds for $\bar{\eta}_{nk}$, $k=1,\ldots,N_n$.

Write $A_1,A_2,\ldots,A_{(r-1)\tilde{\ell}_n}$, $B_1,\ldots,B_{\tilde{\ell}_n}$ for $\{X_1\le u_n\},\ldots,\{X_{\tilde{\ell}_n}\le u_n\}$, $\{X_{\tilde{\ell}_n+\ell_n+1}\le u_n\},\ldots,$ $\{X_{2\tilde{\ell}_n+\ell_n}\le u_n\},\ldots,\{X_{(r-1)(\tilde{\ell}_n+\ell_n)+1}\le u_n\},\ldots,\{X_{r\tilde{\ell}_n+(r-1)\ell_n}\le u_n\}$ respectively and

$$f_t(I_{A_1},\ldots,I_{A_{(r-1)\tilde{\ell}_n}}) = e^{i\sum_{k=1}^{r-1}\bar{\xi}_{nk}t} , \quad g_t(I_{B_1},\ldots,I_{B_{\tilde{\ell}_n}}) = e^{i\bar{\xi}_{nr}t} .$$

Let $f_t(p)$ be the value of the random variable $f_t(I_{A_1},\ldots,I_{A(r-1)\tilde{\ell}_n})$ at such points that p of I_{A_i}, $i=1,\ldots,(r-1)\tilde{\ell}_n$ are equal to 0 and all others are 1 . Then we have

$$Ef_tg_t = f_t(0)g_t(0)P(\bar{A}_1\ldots\bar{A}_{(r-1)\tilde{\ell}_n}\bar{B}_1\ldots\bar{B}_{\tilde{\ell}_n})$$

$$+ f_t(0)\sum_{p=1}^{\tilde{\ell}_n} g_t(p)\sum_{1\le j_1<\ldots<j_p\le\tilde{\ell}_n} P(\bar{A}_1\ldots\bar{A}_{(r-1)\tilde{\ell}_n}\cap B_{j_1\ldots j_p})$$

$$+ g_t(0)\sum_{p=1}^{(r-1)\tilde{\ell}_n} f_t(p)\sum_{1\le j_1<\ldots<j_p\le(r-1)\tilde{\ell}_n} P(A_{j_1\ldots j_p}\cap\bar{B}_1\ldots\bar{B}_{\ell})$$

$$+\sum_{p=1}^{(r-1)\tilde{\ell}_n}\sum_{1\le i_1<\ldots<i_p\le(r-1)\tilde{\ell}_n}\sum_{q=1}^{\tilde{\ell}_n}\sum_{1\le j_1<\ldots<j_q\le\tilde{\ell}_n} f_t(p)g_t(q)P(A_{i_1\ldots i_p}\cap B_{j_1\ldots j_q})$$

Since (1.2) holds (including s=0 or t=0) and $|f_t(p)| = |g_t(p)| = 1$ for any p,q, (1.4) follows from

$$|Ee^{i\sum_{k=1}^{N_n}\xi_{nk}t} - (Ee^{i\bar{\xi}_{n1}t})^{N_n}| \le \sum_{r=1}^{N_n}|Ef_tg_t - Ef_tEg_t|$$

$$\le \alpha(n,\ell_n)\sum_{r=1}^{N_n}\sum_{p=0}^{(r-1)\tilde{\ell}_n}\sum_{q=0}^{\tilde{\ell}_n}\binom{(r-1)\tilde{\ell}_n}{p}\binom{\tilde{\ell}_n}{q}2^{p+q}$$

$$= \sum_{r=1}^{N_n} 3^{r\tilde{\ell}_n} \alpha(n,\ell_n) \le 3^n \alpha(n,\ell_n) .$$

In the same way, we can show that

$$\left| E e^{i \sum_{k=1}^{N_n} \bar{\eta}_{nk} t} - (E e^{i\bar{\eta}_{n1} t})^{N_n} \right| \le \sum_{r=1}^{N_n} 3^{r\ell_n} \alpha(n,\tilde{\ell}_n) \le 3^n \alpha(n,\ell_n) ,$$

completing the proof of the lemma.

Lemma 1.3 If $\lim_n (n/k_n)\cdot F(u_n) = 1$, then

$$(1.5) \quad \left| \sum_{1\le i<j<k\le\tilde{\ell}_n} E\tilde{I}_{ni}\tilde{I}_{nj}\tilde{I}_{nk} \right| \le C_1\tilde{\ell}_n^3\alpha(n,\ell_n) + C_2\frac{\tilde{\ell}_n\ell_n^2 k_n^2}{n^2} + C_3\tilde{\ell}_n\ell_n \left| \sum_{j=1}^{\ell_n-1} E\tilde{I}_{n1}\tilde{I}_{nj+1} \right|$$

where C_1, C_2, C_3 are constants.

Proof: Using stationarity of the process, we obtain

$$\sum_{1\le i<j<k\le\tilde{\ell}_n} E\tilde{I}_{nk}\tilde{I}_{nj}\tilde{I}_{nk} = \sum_{s=1}^{\tilde{\ell}_n-2} \sum_{t=1}^{\tilde{\ell}_n-s-1} (\tilde{\ell}_n-s-t)E\tilde{I}_{n1}\tilde{I}_{ns+1}\tilde{I}_{ns+t+1} .$$

Since

$$E\tilde{I}_{n1}\tilde{I}_{ns+1}\tilde{I}_{ns+t+1} = P(X_1\le u_n, X_{s+1}\le u_n, X_{s+t+1}\le u_n) - F(u_n)[P(X_1\le u_n, X_{s+1}\le u_n)$$
$$+ P(X_1\le u_n, X_{s+t+1}\le u_n) + P(X_{s+1}\le u_n, X_{s+t+1}\le u_n)] + 2F^3(u_n) ,$$

it follows that

$$\left| \sum_{s=\ell_n}^{\tilde{\ell}_n-2} \sum_{t=1}^{\tilde{\ell}_n-s-1} (\tilde{\ell}_n-s-t)E\tilde{I}_{n1}\tilde{I}_{ns+1}\tilde{I}_{ns+t+1} \right|$$

$$\le \sum_{s=\ell_n}^{\tilde{\ell}_n-2} \sum_{t=1}^{\tilde{\ell}_n-s-1} \tilde{\ell}_n\{|F_{1,s+1,s+t+1}(u_n)-F(u_n)F_{s+1,s+t+1}(u_n)|+|F_{1,s+1}(u_n)-F^2(u_n)|$$
$$+ |F_{1,s+t+1}(u_n)-F^2(u_n)|\} \le 3\tilde{\ell}_n^3\alpha(n,\ell_n) ,$$

and in the same way that

$$\left|\sum_{s=1}^{\ell_n-1}\sum_{t=\ell_n}^{\tilde{\ell}_n-s-1}(\tilde{\ell}_n-s-t)E\tilde{I}_{n1}\tilde{I}_{ns+1}\tilde{I}_{ns+t+1}\right| \le 3\tilde{\ell}_n^3\,\alpha(n,\ell_n)\ .$$

Noticing that

$E|\tilde{I}_{n1}\tilde{I}_{ns+1}| = [1-2F(u_n)]E\tilde{I}_{n1}\tilde{I}_{ns+1} + 4F^2(u_n)[1-F(u_n)]^2$, we have

$$\left|\sum_{s=1}^{\ell_n-1}\sum_{t=1}^{\ell_n-1}(\tilde{\ell}_n-s-t)E\tilde{I}_{n1}\tilde{I}_{ns+1}\tilde{I}_{ns+t+1}\right| \le \tilde{\ell}_n\ell_n\sum_{s=1}^{\ell_n-1}E|\tilde{I}_{n1}\tilde{I}_{ns+i}|$$

$$\le 4\tilde{\ell}_n\ell_n^2F^2(u_n)[1-F(u_n)]^2 + \tilde{\ell}_n\ell_n|1-2F(u_n)|\left|\sum_{s=1}^{\ell_n-1}E\tilde{I}_{n1}\tilde{I}_{ns+1}\right|$$

$$\le C_2\frac{\tilde{\ell}_n\ell_n^2k_n^2}{n^2} + C_3\tilde{\ell}_n\ell_n\left|\sum_{j=1}^{\ell_n-1}E\tilde{I}_{n1}\tilde{I}_{nj+1}\right| ,$$

so that the lemma is proved.

Lemma 1.4 If $\lim_n (n/k_n)\cdot F(u_n) = 1$, then

$$\left|\sum_{1\le i\le j\le k\le\ell\le\tilde{\ell}_n}E\tilde{I}_{ni}\tilde{I}_{nj}\tilde{I}_{nk}\tilde{I}_{n\ell}\right| \le C_1\tilde{\ell}_n^4\,\alpha(n,\ell_n) + C_2\tilde{\ell}_n^2\Big(\sum_{s=1}^{\ell_n-1}E\tilde{I}_{n1}\tilde{I}_{ns+1}\Big)^2$$

$$+ C_3\tilde{\ell}_n\ell_n^3k_n^2/n^2 + C_4\tilde{\ell}_n\ell_n^2\left|\sum_{s=1}^{\ell_n-1}E\tilde{I}_{n1}\tilde{I}_{ns+1}\right|$$

where C_1,C_2,C_3,C_4 are constants.

Proof: Using stationarity of the process we obtain

$$\sum_{i<j<k<\ell}E\tilde{I}_{ni}\tilde{I}_{nk}\tilde{I}_{nk}\tilde{I}_{n\ell} = \sum_{s=1}^{\tilde{\ell}_n-3}\sum_{u=1}^{\tilde{\ell}_n-s-2}\sum_{t=1}^{\tilde{\ell}_n-s-u-1}(\tilde{\ell}_n-s-t-u)E\tilde{I}_{n1}\tilde{I}_{ns+1}\tilde{I}_{ns+t+1}\tilde{I}_{ns+t+u+1}\ .$$

Since

$$E\tilde{I}_{n1}\tilde{I}_{ns+1}\tilde{I}_{ns+t+1}\tilde{I}_{ns+t+u+1} = F_{1,s+1,s+t+1,s+t+u+1}(u_n) - F(u_n)[F_{1,s+1,s+t+1}(u_n)$$

$$+ F_{1,s+1,s+t+u+1}(u_n) + F_{1,s+t+1,s+t+u+1}(u_n) + F_{1,t+1,t+u+1}(u_n)]$$

$$+ F^2(u_n)[F_{1,s+1}(u_n)+F_{1,s+t+1}(u_n)+F_{1,s+t+u+1}(u_n)+F_{1,t+1}(u_n)+F_{1,t+u+1}(u_n)$$

$$+ F_{1,u+1}(u_n)] - 3F^4(u_n)\ ,$$

it follows in the same way as in the proof of lemma 1.2 that

$$\Big|\sum_{s=\ell_n}^{\tilde{\ell}_n-3}\ \sum_{u=1}^{\tilde{\ell}_n-s-2}\ \sum_{t=1}^{\tilde{\ell}_n-s-u-1} (\tilde{\ell}_n-s-t-u)E\tilde{I}_{n1}\tilde{I}_{ns+1}\tilde{I}_{ns+t+1}\tilde{I}_{ns+t+u+1}\Big| \le 7\tilde{\ell}_n^4\alpha(n,\ell_n) .$$

$$\Big|\sum_{s=1}^{\ell_n-1}\ \sum_{u=\ell_n}^{\tilde{\ell}_n-s-2}\ \sum_{t=1}^{\tilde{\ell}_n-s-u-1} (\tilde{\ell}_n-s-t-u)E\tilde{I}_{n1}\tilde{I}_{ns+1}\tilde{I}_{ns+t+1}\tilde{I}_{ns+t+u+1}\Big| \le 7\tilde{\ell}_n^4\alpha(n,\ell_n) .$$

Writing

$$\begin{aligned}
E\tilde{I}_{n1}\tilde{I}_{ns+1}\tilde{I}_{ns+t+1}\tilde{I}_{ns+t+u+1} &= [F_{1,s+1,s+t+1,s+t+u+1}(u_n)-F_{1,s+1}(u_n)F_{1,u+1}(u_n)] \\
&- F(u_n)[F_{1,s+1,s+t+1}(u_n)-F(u_n)F_{1,s+1}(u_n)]-F(u_n)[F_{1,s+t+1,s+t+u+1}(u_n)-F(u_n)F_{1,u+1}(u_n)] \\
&+ F^2(u_n)[F_{1,s+t+1}(u_n)-F^2(u_n)] + F^2(u_n)[F_{1,s+t+u+1}(u_n)-F^2(u_n)] \\
&+ F^2(u_n)[F_{1,t+1}(u_n)-F^2(u_n)] + F^2(u_n)[F_{1,t+u+1}(u_n)-F^2(u_n)] \\
&- F(u_n)[F_{1,s+1,s+t+u+1}(u_n) - F(u_n)F_{1,s+1}(u_n)] \\
&- F(u_n)[F_{1,t+1,t+u+1}(u_n)-F(u_n)F_{1,u+1}(u_n)] + [F_{1,s+1}(u_n)-F^2(u_n)][F_{1,u+1}(u_n)-F^2(u_n)] ,
\end{aligned}$$

we also have

$$\Big|\sum_{s=1}^{\ell_n-1}\ \sum_{u=1}^{\ell_n-1}\ \sum_{t=\ell_n}^{\ell_n-s-u-1} (\tilde{\ell}_n-s-t-u)E\tilde{I}_{n1}\tilde{I}_{ns+1}\tilde{I}_{ns+t+1}\tilde{I}_{ns+t+u+1}\Big| \le 9\tilde{\ell}_n^4\alpha(n,\ell_n)+\tilde{\ell}_n^2\Big(\sum_{s=1}^{\ell_n-1} E\tilde{I}_{n1}\tilde{I}_{ns+1}\Big)^2$$

Finally we can show that

$$\Big|\sum_{s=1}^{\ell_n-1}\ \sum_{t=1}^{\ell_n-1}\ \sum_{u=1}^{\ell_n-1} (\tilde{\ell}_n-s-t-u)E\tilde{I}_{n1}\tilde{I}_{ns+1}\tilde{I}_{ns+t+1}\tilde{I}_{ns+t+u+1}\Big| \le C_4\tilde{\ell}_n\ell_n^2\Big|\sum_{s=1}^{\ell_n-1} E\tilde{I}_{n1}\tilde{I}_{s+1}\Big| + C_5\tilde{\ell}_n\ell_n^3k_n^2/n^2 .$$

Hence the lemma is proved.

§2. Some limit theorems

We introduce the following assumptions:

Assumption I: For some sequence $\{\ell_n\}$ of positive integers,

(2.1) $(n/k_n^{1/2})\cdot\beta(n,\ell_n) \to 0 \quad (n \to \infty)$

$$(2.2)\quad \lim_n (n/k_n)\cdot\sum_{j=1}^{[k_n^{1/2}]-1} E\tilde{I}_{n1}\tilde{I}_{nj+1} = \sigma$$

$$(2.3)\quad \lim_n (n/k^{3/2})\cdot\sum_{j=1}^{[k_n^{1/2}]-1} jE\tilde{I}_{n1}\tilde{I}_{nj+1} = 0 .$$

Assumption II. For some sequence $\{\ell_n\}$ of positive integers,

$$(2.4)\quad 3^n\alpha(n,\ell_n) \to 0 \quad (n \to \infty)$$

and (2.2), (2.3) hold.

It is obvious that the constant σ in (2.2) may be different if the sequence $\{u_n\}$ is changed. But we can show that σ must be the same for any $\{u_n\}$ satisfying (1.1) with some real u.

Lemma 2.1 If (2.1) or (2.4) holds for some $\ell_n = o(k_n^{1/2})$, $\ell_n \to \infty$, then (2.2) and (2.3) hold, if and only if

$$(2.2)'\quad \lim_n (n/k_n)\cdot\sum_{j=1}^{\ell_n-1} E\tilde{I}_{n1}\tilde{I}_{nj+1} = \sigma$$

$$(2.3)'\quad \lim_n (n/k_n^{3/2})\cdot\sum_{j=1}^{\ell_n-1} jE\tilde{I}_{n1}\tilde{I}_{nj+1} = 0.$$

Furthermore, if (1.1) holds for some $u \in R$, we can use

$$(2.2)''\quad \lim_n (n/k_n)\cdot\sum_{j=1}^{\ell_n-1} [P(X_1 \le a(k_n/n),\ X_{j+1} \le a(k_n/n)) - (k_n/n)^2] = \sigma$$

$$(2.3)''\quad \lim_n (n/k_n^{3/2})\cdot\sum_{j=1}^{\ell_n-1} j[P(X_1 \le a(k_n/n),\ X_{j+1} \le a(k_n/n)) - (k_n/n)^2] = 0$$

instead of (2.2)' and (2.3)' respectively in the above statements. In (2.2)" and (2.3)",

$$a(k_n/n) = \begin{cases} a_n - 0 & \text{if } F(a_n) - k_n/n \ge k_n/n - F(a_n-0) \\ a_n & \text{if } F(a_n) - k_n/n < k_n/n - F(a_n-0) \ , \end{cases}$$

where a_n is a real number such that $F(a_n-0) \le k_n/n \le F(a_n)$, and the event $\{X_j \le a_n - 0\}$ is defined as $\{X_j < a_n\}$.

Proof: The first part of the lemma follows from

$$|(n/k_n) \cdot \sum_{j=\ell_n}^{[k_n^{1/2}]-1} E\tilde{I}_{n1}\tilde{I}_{nj+1}| < (n/k_n) \cdot [k_n^{1/2}]\alpha(n,\ell_n) \to 0$$

$$|(n/k_n^{3/2}) \cdot \sum_{j=\ell_n}^{[k_n^{1/2}]-1} jE\tilde{I}_{n1}\tilde{I}_{nj+1}| \le (n/k_n^{3/2}) \cdot [k_n^{1/2}]^2\alpha(n,\ell_n) \to 0 \ .$$

Now we show the second part. By the definition of $a(k_n/n)$, it is easy to see that $|F(a(k_n/n)) - k_n/n| \le |F(x) - k_n/n|$ for any x. Therefore we have

$$|(n/k_n) \cdot \sum_{j=1}^{\ell_n-1} E\tilde{I}_{n1}\tilde{I}_{nj+1} - (n/k_n) \cdot \sum_{j=1}^{\ell_n-1} [P(X_1 \le a(k_n/n),\ X_{j+1} \le a(k_n/n)) - (k_n/n)^2]|$$

$$\le (n/k_n) \cdot \sum_{j=1}^{\ell_n-1} [|P(X_1 \le u_n,\ X_{j+1} \le u_n) - P(X_1 \le a(k_n/n),\ X_{j+1} \le a(k_n/n))| + |F^2(u_n) - (k_n/n)^2|]$$

$$\le \ 2(n/k_n) \cdot \sum_{j=1}^{\ell_n-1} [|F(u_n) - F(a(k_n/n))| + |F(u_n) - k_n/n|]$$

$$\le 6(\ell_n/k_n^{1/2}) \cdot (n/k_n^{1/2}) \cdot |F(u_n) - k_n/n| \to 0 \ .$$

This proves that (2.2)" and (2.2)' are equivalent. In the same way, we can show that (2.3)" is equivalent to (2.3)'.

Let $$\bar{S}_n = \sum_{j=1}^{n} \bar{I}_{nj} = \sum_{k=1}^{N_n} \bar{\xi}_{nk} + \sum_{k=1}^{N_n} \bar{\eta}_{nk} + \bar{\zeta}_n \ .$$

We now start to discuss the limiting distribution of $\bar{S}_n$.

Lemma 2.2 If assumption I or II holds for some $\ell_n = o(k_n^{1/2})$ and

(2.5) $\lim_n nF(u_n)/k_n = 1,$

then

(2.6) $P(\bar{S}_n \le x) \overset{c}{\to} \Phi_\lambda(x)$,

if and only if

(2.7) $\lim_n (Ee^{i\bar{\xi}_{n1} t})^{N_n} = \psi(t)$.

When (2.6) or (2.7) holds, we have

(2.8) $\psi(t) = \int e^{itx}\, d\Phi_\lambda(x).$

Proof: Let $\tilde{\ell}_n = [k_n^{1/2}]$. If $n - N_n(\ell_n+\tilde{\ell}_n)<\ell_n$, we have

$$0 \le E\bar{\zeta}_n^2 \le \ell_n^2/k_n \to 0 \ .$$

If $n - N_n(\ell_n+\tilde{\ell}_n) \ge \ell_n$, we also have

$$0 \le E\bar{\zeta}_n^2 = \frac{1}{k_n}\{[n-N_n(\ell_n+\tilde{\ell}_n)]F(u_n)[1-F(u_n)] + 2\sum_{j=1}^{n-N_n(\ell_n+\tilde{\ell}_n)} [n-N_n(\ell_n+\tilde{\ell}_n)-j]E\tilde{I}_{n1}\tilde{I}_{nj+1}\}$$

$$\le C/N_n + \frac{2}{N_n}\left|\frac{n}{k_n}\sum_{j=1}^{\tilde{\ell}_n-1} E\tilde{I}_{n1}\tilde{I}_{nj+1}\right| + \frac{2}{k_n}\left|\sum_{j=1}^{\ell_n-1} jE\tilde{I}_{n1}\tilde{I}_{nj+1}\right| + 2\frac{(\tilde{\ell}_n+\ell_n)^2}{k_n}\alpha(n,\ell_n) \to 0 \ .$$

Hence by Chebyshev's inequality, it follows that

$$\bar{\zeta} \to 0 \quad [P] \ .$$

Since $E\bar{\eta}_{n1} = 0$, we have

$$N_n|Ee^{i\eta_{n1}t}-1| \le N_n\, E\bar{\eta}_{n1}^2 \cdot t^2/2 = (t^2/2)\cdot(N_n/k_n)\{\ell_n F(u_n)[1-F(u_n)] + 2\sum_{j=1}^{\ell_n-1}(\ell_n-j)E\tilde{I}_{n1}\tilde{I}_{nj+1}\}$$

$$\cdot \frac{t^2}{2}\left\{\frac{\ell_n}{\tilde{\ell}_n}\frac{n}{k_n}F(u_n)[1-F(u_n)]+2\frac{\ell_n}{\tilde{\ell}_n}\left|\frac{n}{k_n}\sum_{j=1}^{\ell_n-1}E\tilde{I}_{n1}\tilde{I}_{nj+1}\right|+2\left|\frac{n}{k_n^{3/2}}\sum_{j=1}^{\ell_n-1}jE\tilde{I}_{n1}\tilde{I}_{nj+1}\right|\right\} \to 0 ,$$

i.e. $Ee^{i\bar{\eta}_{n1}t} = 1 + 0(\frac{1}{N_n})$. Therefore $\lim_n (Ee^{i\bar{\eta}_{n1}t})^{N_n} = 1$. By using lemma 1.2, this implies that

$$\lim_n Ee^{i\sum_{k=1}^{N_n}\bar{\eta}_{nk}t} = 1, \text{ i.e.}$$

$$\sum_{k=1}^{N_n}\bar{\eta}_{nk} \to 0[P].$$

From the above argument, it follows that (2.6) is equivalent to $P(\sum_{k=1}^{N_n}\bar{\xi}_{nk} \le x) \xrightarrow{c} \phi_\lambda(x)$, i.e.

$$\lim_n Ee^{i\sum_{k=1}^{N_n}\bar{\xi}_{nk}t} = \psi(t) ,$$

where $\psi(t)$ is defined by (2.8). Using lemma 1.2 again, we see that (2.6) and (2.7) are equivalent. Hence the lemma is proved.

Lemma 2.3 If assumption I or II holds for some $\ell_n = o(\min(k_n^{1/2}, \frac{n^{1/2}}{k_n^{1/4}}))$, and (2.5) holds, then

$$\lim_n (Ee^{i\bar{\xi}_{n1}t})^{N_n} = e^{-[\sigma+\frac{1}{2}(1-\lambda)]t^2}$$

Proof: From Taylor's formula, it is easy to show that

$$|e^{it} - \sum_{k=0}^{n}\frac{(it)^k}{k!}| \le \frac{|t|^{n+1}}{(n+1)!} , \quad n=0,1,2,\ldots, \ \forall\, t$$

Therefore, taking n=3, we have

$$|Ee^{i\bar{\xi}_{n1}t} - 1 - itE\bar{\xi}_{n1} - \frac{(it)^2}{2!}E\bar{\xi}_{n1}^2 - \frac{(it)^3}{3!}E\bar{\xi}_{n1}^3|$$

$$\leq E\left|e^{i\bar{\xi}_{n1}t}-1-it\bar{\xi}_{n1}-\frac{(it)^2}{2!}\bar{\xi}_{n1}^2-\frac{(it)^3}{3!}\bar{\xi}_{n1}^3\right| < \frac{|t|^4E\bar{\xi}_{n1}^4}{4!} .$$

Noticing that $E\bar{\xi}_{n1} = 0$, the lemma can be proved if we show that

$$E\bar{\xi}_{n1}^2 = \frac{1}{N_n}[(1-\lambda)+2\sigma] + o(\frac{1}{N_n}) ,$$

$$E\bar{\xi}_{n1}^3 = o(\frac{1}{N_n}) , \quad E\bar{\xi}_{n1}^4 = o(\frac{1}{N_n}) ,$$

which is equivalent to

(2.9) $N_nE\bar{\xi}_{n1}^2 \to (1-\lambda) + 2\sigma$,

(2.10) $N_nE\bar{\xi}_{n1}^3 \to 0$,

(2.11) $N_nE\bar{\xi}_{n1}^4 \to 0$.

Since under assumption I or II,

$$N_nE\bar{\xi}_{n1}^2 = \frac{N_n}{k_n}\{\tilde{\ell}_nF(u_n)[1-F(u_n)] + 2\sum_{j=1}^{\tilde{\ell}_n-1}(\tilde{\ell}_n-j)E\tilde{I}_{n1}\tilde{I}_{nj+1}\}$$

$$= \frac{N_n\tilde{\ell}_nF(u_n)[1-F(u_n)]}{k_n} + \frac{2N_n\tilde{\ell}_n}{k_n}\sum_{j=1}^{\tilde{\ell}_n-1}E\tilde{I}_{n1}\tilde{I}_{nj+1} - \frac{2N_n}{k_n}\sum_{j=1}^{\tilde{\ell}_n-1}jE\tilde{I}_{n1}\tilde{I}_{nj+1} \to (1-\lambda) + 2\sigma ,$$

(2.9) is obvious. To prove (2.10), we expand

(2.12) $$N_nE\bar{\xi}_{n1}^3 = \frac{N_n}{k_n^{3/2}}\sum_{j=1}^{\tilde{\ell}_n}E\tilde{I}_{nj}^3 + \frac{3N_n}{k_n^{3/2}}\sum_{i\neq j}E\tilde{I}_{ni}^2\tilde{I}_{nj} + \frac{6N_n}{k_n^{3/2}}\sum_{i<j<k}E\tilde{I}_{ni}\tilde{I}_{nj}\tilde{I}_{nk} .$$

For the first term on the right hand of (2.12), we have

$$\left|\frac{N_n}{k_n^{3/2}}\sum_{j=1}^{\tilde{\ell}_n}E\tilde{I}_{nj}^3\right| \leq \frac{N_n}{k_n}|E\tilde{I}_{nj}^3| \leq \frac{1}{k_n^{1/2}}|1-2F(u_n)|\cdot\frac{n}{k_n}F(u_n)[1-F(u_n)] \to 0 .$$

Noticing that $E\tilde{I}_{ni}^2\tilde{I}_{nj} = [1-2F(u_n)]E\tilde{I}_{ni}\tilde{I}_{nj}$, we have

$$\sum_{i\neq j} E\tilde{I}^2_{ni}\tilde{I}_{nj} = 2[1-F(u_n)]\sum_{j=1}^{\tilde{\ell}_n-1}(\tilde{\ell}_n-j)E\tilde{I}_{n1}\tilde{I}_{nj+1} .$$

Therefore, for the second term on the right hand of (2.12), it follows that

$$\left|\frac{3N_n}{k_n^{3/2}}\sum_{i\neq j} E\tilde{I}^2_{ni}\tilde{I}_{nj}\right| \le \frac{C}{k_n^{1/2}}\left|\frac{N_n}{k_n}\sum_{j=1}^{\tilde{\ell}_n-1}(\tilde{\ell}_n-j)E\tilde{I}_{n1}\tilde{I}_{nj+1}\right| \to 0 .$$

Lastly, by using lemma 1.3, we obtain

$$\left|\frac{6N_n}{k_n^{3/2}}\sum_{i<j<k} E\tilde{I}_{ni}\tilde{I}_{nj}\tilde{I}_{nk}\right| \le C_1\frac{\tilde{\ell}^2_n}{k_n}\cdot\frac{n}{k_n^{1/2}}\alpha(n,\ell_n)+C_2\frac{k_n^{1/2}\ell_n^2}{n}+C_3\frac{\ell_n}{k_n^{1/2}}\cdot\frac{n}{k_n}\left|\sum_{j=1}^{\ell_n-1}E\tilde{I}_{n1}\tilde{I}_{nj+1}\right| \to 0.$$

Hence (2.10) holds. To prove (2.12), expand

$$(2.13)\quad N_nE\bar{\xi}^4_{n1} = \frac{N_n\tilde{\ell}_n}{k_n^2}E\tilde{I}^4_{n1} + \frac{4N_n}{k_n^2}\sum_{i\neq j}E\tilde{I}^3_{ni}\tilde{I}_{nj} + \frac{6N_n}{k_n^2}\sum_{i<j}E\tilde{I}^2_{ni}\tilde{I}^2_{nj}$$

$$+\frac{12N_n}{k_n^2}\sum_{\substack{j\neq i,k\neq i\\ j<k}}E\tilde{I}^2_{ni}\tilde{I}_{nj}\tilde{I}_{nk} + \frac{24N_n}{k_n^2}\sum_{i<j<k<\ell}E\tilde{I}_{ni}\tilde{I}_{nj}\tilde{I}_{nk}\tilde{I}_{n\ell} .$$

For the first term on the right hand of (2.13), it follows that

$$\frac{N_n\tilde{\ell}_n}{k_n^2}E\tilde{I}^4_{n1} = \frac{N_n\tilde{\ell}_n}{k_n^2}F(u_n)[1-F(u_n)][1-3F(u_n)+3F^2(u_n)] \le \frac{C}{k_n}\to 0 .$$

Since $E\tilde{I}^3_{ni}\tilde{I}_{nj} = [1-3F(u_n)+3F^2(u_n)]E\tilde{I}_{ni}\tilde{I}_{nj}$, for the second term, we also have

$$\left|\frac{N_n}{k_n^2}\sum_{i\neq j}E\tilde{I}^3_{ni}\tilde{I}_{nj}\right| \le \frac{C}{k_n}\left|\frac{N_n}{k_n}\sum_{j=1}^{\tilde{\ell}_n-1}(\tilde{\ell}_n-j)E\tilde{I}_{n1}\tilde{I}_{nj+1}\right| \to 0 .$$

By using $E\tilde{I}^2_{ni}\tilde{I}^2_{nj} = F^2(u_n)[1-F(u_n)]^2 + [1-2F(u_n)]^2E\tilde{I}_{ni}\tilde{I}_{nj}$, it is seen that

$$\frac{N_n}{k_n^2}\sum_{i<j}E\tilde{I}^2_{ni}\tilde{I}^2_{nj} \le \frac{N_n\tilde{\ell}^2_n}{k_n^2}F^2(u_n)[1-F(u_n)]^2 + \frac{N_nC}{k_n^2}\left|\sum_{j=1}^{\tilde{\ell}_n-1}(\tilde{\ell}_n-j)E\tilde{I}_{n1}\tilde{I}_{nj+1}\right| \to 0 .$$

Noticing that $E\tilde{I}_{ni}^2\tilde{I}_{nj}\tilde{I}_{nk} = [1-2F(u_n)]E\tilde{I}_{ni}\tilde{I}_{nj}\tilde{I}_{nk} + 2F^2(u_n)[1-F(u_n)]E\tilde{I}_{nj}\tilde{I}_{nk}$ we obtain, for the fourth term,

$$\left|\frac{N_n}{k_n^2}\sum_{\substack{j\neq i,k\neq i\\ j<k}} E\tilde{I}_{ni}^2\tilde{I}_{nj}\tilde{I}_{nk}\right| \le \frac{C_1}{k_n^{1/2}}\left|\frac{N_n}{k_n^{3/2}}\sum_{i<j<k} E\tilde{I}_{ni}\tilde{I}_{nj}\tilde{I}_{nk}\right| + \frac{C_2}{N_n}\cdot\frac{k_n}{n}\left|\frac{N_n}{k_n}\sum_{j=1}^{\tilde{\ell}_n-1}(\tilde{\ell}_n-j)E\tilde{I}_{n1}\tilde{I}_{nj+1}\right| \cdot n$$

Lastly, by lemma 1.4, it follows that $\left|\frac{N_n}{k_n^2}\sum_{i<j<k<\ell} E\tilde{I}_{ni}\tilde{I}_{nj}\tilde{I}_{nk}\tilde{I}_{n\ell}\right|$

$$C_1\frac{\tilde{\ell}_n^3}{k_n^{3/2}}\cdot\frac{n}{k_n^{1/2}}\alpha(n,\ell_n)+C_2\frac{N_n\tilde{\ell}_n^2}{k_n^2}\left|\sum_{s=1}^{\ell_n-1}E\tilde{I}_{n1}\tilde{I}_{ns+1}\right|^2+C_3\frac{\ell_n^2k_n^{1/2}}{n}\cdot\frac{\tilde{\ell}_n}{k_n^{1/2}}+C_1\frac{\ell_n^2 n}{k_n^3}\sum_{s=1}^{\ell_n-1}E\tilde{I}_{n1}\tilde{I}_{ns+1}$$

Hence (2.12) holds, and the lemma is proved.

From lemma 2.2 and 2.3, we obtain

Theorem 2.4 If assumption I or II holds for some $\ell_n = o(\min(k_n^{1/2}, \frac{n^{1/2}}{k_n^{1/4}}))$, and (2.5) holds, then (2.6) holds with

$$\Phi_\lambda(x) = \frac{1}{(2\pi)^{1/2}\sigma_\lambda}\int_{-\infty}^{x} \exp(-\frac{t^2}{2\sigma_\lambda^2})\,dt = \Phi(\frac{x}{\sigma_\lambda}) ,$$

where $\sigma_\lambda^2 = (1-\lambda)+2\sigma$, $\Phi(x)$ is the normal distribution function with mean 0 and variance 1, and when $\sigma_\lambda = 0$, $\Phi(\frac{x}{\sigma_\lambda})$ is defined to be 1 for $x\geq 0$ and 0 for $x<0$. The above statement is still true if (2.2)', (2.3)' are used instead of (2.2), (2.3) in assumption I and II.

Furthermore, using lemma 2.1, we obtain

Theorem 2.5 If (1.1) holds, and assumption I or II holds for some $\ell_n = o(\min(k_n^{1/2}, \frac{n^{1/2}}{k_n^{1/4}}))$, then the conclusion of Theorem 2.4 follows. The conclusion of Theorem 2.4 is still true if (2.2)", (2.3)" are used instead of (2.2), (2.3) in assumptions I and II.

It is easy to show that if for some $\ell_n = o(k_n^{1/2})$

(2.14) $$\lim_{n\to\infty} \frac{n}{k_n} \sum_{j=1}^{\ell_n - 1} |E\tilde{I}_{n1}\tilde{I}_{nj+1}| = 0 ,$$

then (2.2) and (2.3) hold with $\sigma=0$. Therefore we obtain

Theorem 2.6 If (2.14) and one of (2.1) and (2.4) holds for some $\ell_n = o(\min(k_n^{1/2}, \frac{n^{1/2}}{k_n^{1/4}}))$, then (2.6) holds with $\Phi_\lambda(x) = \Phi(\frac{x}{(1-\lambda)^{1/2}})$.

§3. The results for general stationary processes

An i.i.d. sequence $\{\hat{X}_n\}$ is called the associated independent sequence of a stationary sequence $\{X_n\}$ if $\hat{X}_n$ has the same marginal d.f. F(x) as X_n. Smirnov [6] has shown that there are constants $a_n>0$, b_n such that

(3.1) $$P(\hat{X}_{k_n}^{(n)} \le a_n x + b_n) \overset{c}{\to} \Psi(x)$$

if and only if

(3.2) $$\frac{n}{k_n^{1/2}}[F(a_n x+b_n) - \frac{k_n}{n}] \overset{c}{\to} (1-\lambda)^{1/2}u(x)$$

where u(x) is a nondecreasing, right continuous, (finite or infinite valued) real function such that $u(-\infty) = \lim_{x\to-\infty} u(x) = -\infty$, $u(\infty) = \lim_{x\to\infty} u(x) = \infty$. The relation between $\Psi(x)$ and u(x) is

(3.3) $\Psi(x) = \Phi(u(x))$.

In this paper, we will find the limiting distribution of $X_{k_n}^{(n)}$ under condition (3.2) considering only the case in which $\Phi(u(x))$ is not degenerate.

Theorem 3.1 Suppose that

1. there are $a_n>0$, b_n such that (3.2) holds with a continuous u(x),
2. for any $u_n = a_n x+b_n$, $x\in B(u(\cdot)) \equiv \{x: |u(x)|<\infty\}$, assumption I or II holds

with some $\ell_n = o(\min(k_n^{1/2}, \frac{n^{1/2}}{k_n^{1/4}}))$. Then the real σ in (2.2) is independent of x

and determined by (2.2)", and

(3.4) $$P(X_{k_n}^{(n)} \le a_n x + b_n) \stackrel{\subset}{\to} \Phi\left(\frac{(1-\lambda)^{1/2}}{\sigma_\lambda} u(x)\right), \quad \sigma_\lambda > 0 .$$

Proof: According to theorem 2.5, we have

$$P(X_{k_n}^{(n)} \le a_n x + b_n) = P\left(\bar{S}_n \ge \frac{n}{k_n^{1/2}} \left[\frac{k_n}{n} - F(a_n x + b_n)\right]\right)$$

$$\to 1 - \Phi\left(-\frac{(1-\lambda)^{1/2}}{\sigma_\lambda} u(x)\right) = \Phi\left(\frac{(1-\lambda)^{1/2}}{\sigma_\lambda} u(x)\right)$$

for all $x \in B(u(\cdot))$. If $u(x) = +\infty$, then $x \ge x_o \equiv \sup\{x: \ u(x) < \infty\}$. By taking $x_n \in B(u(\cdot))$, $x_n \uparrow x_o$ and using the continuity of $\Phi(\cdot)$ and $u(\cdot)$, it follows that

$$\underline{\lim_n}\, P(X_{k_n}^{(n)} \le a_n x + b_n) \ge \underline{\lim_n}\, P(X_{k_n}^{(n)} \le a_n x_o + b_n) \ge \lim_n P(X_{k_n}^{(n)} \le a_n x_n + b_n)$$

$$= \lim_n \Phi\left\{\frac{(1-\lambda)^{1/2}}{\sigma_\lambda} u(x_n)\right) = 1 ,$$

i.e. $\lim_n P(X_{k_n}^{(n)} \le a_n x + b_n) = \Phi\left(\frac{(1-\lambda)^{1/2}}{\sigma_\lambda} u(x)\right)$ still holds. Similarly we can show (3.4) also holds if $u(x) = -\infty$. This proves the theorem.

From this theorem we know that under assumption I or II, the limiting distributions of $\frac{X_{k_n}^{(n)} - b_n}{a_n}$ and $\frac{\hat{X}_{k_n}^{(n)} - b_n}{a_n}$ may be different. In fact, we have

Theorem 3.2 If there are $a_n > 0$, b_n such that (3.2) holds with a continuous $u(x)$, (2.14) and either (2.1) or (2.4) holds with some $\ell_n = o(\min(k_n^{1/2}, \frac{n^{1/2}}{k_n^{1/4}}))$ for any $u_n = a_n x + b_n$, $x \in B(u(\cdot))$, then

(3.5) $$P(X_{k_n}^{(n)} \le a_n x + b_n) \to \Phi(u(x)) .$$

Proof: Notice (2.14) implies (2.2) and (2.3) with $\sigma = 0$.

Theorem 3.3 If in theorem 3.1 or 3.2, $\{k_n\}$ is nondecreasing and $\lambda=0$, then (3.4) and (3.5) hold respectively.

Proof: It is proved by Wu [8] that if $\{k_n\}$ is nondecreasing then the only possible types of limiting laws of $\{\hat{X}_{k_n}^{(n)}\}$ are $\Phi(u_i(x))$, $i=1,2,3$, where

$$u_1(x) = \begin{cases} -\alpha\log|x| & x<0 \quad (\alpha>0) \\ \infty & x\geq 0 \end{cases}$$

$$u_2(x) = \begin{cases} -\infty & x\leq 0 \\ \alpha\log x & x>0 \quad (\alpha>0) \end{cases}$$

$$u_3(x) = x \ .$$

The theorem follows from theorem 3.1 and 3.2, by noting that $u_i(x)$, $i=1,2,3$ are continuous.

Smirnov [6] has shown that $u(x)$ satisfying (3.2) need not be continuous if $0<\lambda<1$ in (0.1). Therefore theorem 3.1 and 3.2 cannot be applied to this case, and a special discussion is therefore needed.

Lemma 3.4 If $\lambda\epsilon(0,1)$ in (0.1) and $\beta(n,\ell_n)$ (or $3^n\alpha(n,\ell_n)$) tend to zero for some $\ell_n = o(k_n^{1/2})$, $\ell_n\to\infty$, then

$$\frac{1}{k_n}\sum_{j=1}^{N}\tilde{I}_{nj} \to 0 \quad [P] \ .$$

Proof: Let $\tilde{\ell}_n = \max(n\beta^{1/2}(n,\ell_n),\ [k_n^{1/2}])$ and

$$\tilde{\xi}_{ni} = \sum_{j=(i-1)(\ell_n+\tilde{\ell}_n)+1}^{(i-1)(\tilde{\ell}_n+\ell_n)+\tilde{\ell}_n}\tilde{I}_{nj} \ , \qquad \tilde{\eta}_{ni} = \sum_{j=(i-1)(\ell_n+\tilde{\ell}_n)+\tilde{\ell}_n+1}^{i(\ell_n+\tilde{\ell}_n)}\tilde{I}_{nj} \ ,$$

$$\tilde{\zeta}_n = \sum_{j=N_n(\ell_n+\tilde{\ell}_n)+1}^{n}\tilde{I}_{nj} \ .$$

We have

$$\frac{1}{k_n^2} E\tilde{\zeta}_n^2 \le \frac{1}{k_n^2} (\tilde{\ell}_n+\ell_n)^2 \to 0.$$

$$\frac{1}{k_n^2} E\Big(\sum_{i=1}^{N_n} \tilde{\eta}_{ni}\Big)^2 \le \frac{1}{k_n^2} (N_n\ell_n)^2 \le \frac{n^2}{k_n^2} \cdot \Big(\frac{\ell_n}{\tilde{\ell}_n}\Big)^2 \to 0 \quad ,$$

so that $\frac{1}{k_n^2} \tilde{\zeta}_n \to 0$ [P], $\frac{1}{k_n} \sum_{i=1}^{N_n} \tilde{\eta}_{ni} \to 0$ [P]. Noticing that under the conditions of the lemma,

$$\Big|Ee^{it\sum_{j=1}^{N_n} \frac{\tilde{\xi}_{nj}}{k_n}} - \Big(Ee^{it\frac{\tilde{\xi}_{n1}}{k_n}}\Big)^{N_n}\Big| \le \beta^{1/2}(n,\ell_n) \to 0 \ ,$$

and that

$$\frac{N_n}{k_n^2} E|\tilde{\xi}_{n1}|^2 \le \frac{N_n}{k_n^2} \tilde{\ell}_n^2 \le \frac{n}{k_n} \cdot \frac{\tilde{\ell}_n}{k_n} \to 0 \ , \text{ we obtain}$$

$$\lim_n Ee^{i\frac{1}{k_n}\sum_{j=1}^{N_n} \tilde{\xi}_{nj}t} = \lim_n \Big(Ee^{i\frac{\tilde{\xi}_{n1}}{k_n}t}\Big)^{N_n} = 1 \ ,$$

and hence $\frac{1}{k_n} \sum_{i=1}^{N_n} \tilde{\xi}_{ni} \to 0$ [P]. This proves the lemma.

Lemma 3.5 Under the conditions of lemma 3.4, if for some real sequence $\{u_n\}$,

$$0 < \underline{\lim}_n P(X_{k_n}^{(n)} \le u_n) \le \overline{\lim}_n P(X_{k_n}^{(n)} \le u_n) < 1 \ ,$$

then (2.5) holds.

Proof: If (2.5) does not hold, from Lemma 3.4 and the fact

$$P(X_{k_n}^{(n)} \le u_n) = P\Big(\frac{1}{k_n} \sum_{j=1}^{n} \tilde{I}_{nj} \ge 1 - \frac{n}{k_n} F(u_n)\Big) \ ,$$

we know that one of the two equations

$$\underline{\lim}_n P(X_{k_n}^{(n)} < u_n) = 0 \quad , \quad \overline{\lim}_n P(X_{k_n}^{(n)} \le u_n) = 1$$

must hold, contradicting the assumption of the lemma 3.5. Hence (2.5) must hold.

Theorem 3.6 Suppose that

1. $\lambda\epsilon(0,1)$ in (0.1);

2. there are $a_n>0$, b_n such that (3.2) holds;

3. for any $u_n = a_n x+b_n$, $x\epsilon B_1(u(\cdot)) \equiv \{x: |u(x)| < \infty$, x is a continuity point of $u(x)\}$, assumption I or II holds for some $\ell_n = o(k_n^{1/4})$, $\ell_n \to \infty$. Then the real σ in (2.2) is independent of x and determined by (2.2)", and (3.4) holds. Furthermore, if conditions 1,2,(2.14) hold and either (2.1) or (2.4), with $\ell_n=o(k_n^{1/4})$ $\ell_n \to \infty$, then (3.5) holds.

Proof: It follows from theorem 2.5 that (3.4) holds for all $x\epsilon B_1(u(\cdot))$. Thus it is sufficient to show that

$$(3.6)\quad \lim_n P(X_{k_n}^{(n)} \le a_n x+b_n) = 1 \quad , \quad \text{if } u(x) = \infty$$

$$(3.7)\quad \lim_n P(X_{k_n}^{(n)} \le a_n x+b_n) = 0 \quad , \quad \text{if } u(x) = -\infty .$$

If (3.6) is not true, we can choose a subsequence such that

$$\lim_{n'} P(X_{k_{n'}}^{(n')} \le a_{n'} x_0+b_{n'}) = \ell < 1$$

for some x_0, $u(x_0) = \infty$. Taking $x_1 \epsilon B_1(u(\cdot))$, we have $x_1<x_0$, and therefore

$$0 < \lim_{n'} P(X_{k_{n'}}^{(n')} \le a_{n'} x_1+b_{n'}) \le \lim_{n'} P(X_{k_{n'}}^{(n')} \le a_{n'} x_0+b_{n'}) = \ell < 1 .$$

According to lemma 3.5, this implies (2.5) with $u_{n'} = a_{n'} x_0+b_{n'}$. By using theorem 2.4, it follows that

$$\Phi\left(\frac{(1-\lambda)^{1/2}}{\sigma_\lambda} u(x_0)\right) = \lim_{n'} \Phi\left(\frac{1}{\sigma_\lambda} \frac{n'}{k_{n'}} \left[F(u_{n'}') - \frac{k_{n'}}{n'}\right]\right)$$

$$= \lim_{n'} P(X_{k_{n'}}^{(n')} \le u_{n'}) = \ell\epsilon(0,1) .$$

This is contrary to $\Phi(\frac{(1-\lambda)^{1/2}}{\sigma_\lambda} u(x_0)) = \Phi(\infty) = 1$. Hence (3.6) must hold, and in a similar way, we can show (3.7). Then the theorem follows.

If the rank sequence $\{k_n\}$ satisfies

(3.8) $n^{1/2}(\frac{k_n}{n} - \lambda) \to t, \ -\infty<t<\infty \ , \ 0<\lambda<1,$

Smirnov [6] has shown that the only possible non-degenerate types of limiting laws of $\{\hat{X}^{(n)}_{k_n}\}$ are $\Phi(\tilde{u}_i(x) - \frac{t}{\tilde{\sigma}_\lambda})$, i=1,2,3,4, where

(3.9)

$$\tilde{u}_1(x) = \begin{cases} -\infty & x\le 0 \\ Cx^\alpha & x\ge 0 \qquad (C>0,\ \alpha>0) \end{cases}$$

$$\tilde{u}_2(x) = \begin{cases} -C|x|^\alpha & x<0 \qquad (C>0,\ \alpha>0) \\ \infty & x\ge 0 \end{cases}$$

$$\tilde{u}_3(x) = \begin{cases} -C_1|x|^\alpha & x<0 \\ C_2x^\alpha & x\ge 0 \qquad (C_1,C_2>0,\ \alpha>0) \end{cases}$$

$$\tilde{u}_4(x) = \begin{cases} -\infty & x<-1 \\ 0 & -1\le x<1 \\ \infty & x\ge 1 \end{cases}$$

and $\tilde{\sigma}_\lambda = [\lambda(1-\lambda)]^{1/2}$. For stationary processes, similar results are obtained as follows.

Theorem 3.7 If (3.8) holds, then under conditions 2 and 3 of theorem 3.6, (3.4) holds, and u(x) in (3.4) is one of the four types

$$u(x) = \tilde{u}_i(x) - \frac{t}{\tilde{\sigma}_\lambda} \qquad i=1,2,3,4 ,$$

and the real σ in (3.4) can be found as follows

(3.10) $$\sigma = \frac{1}{\lambda} \sum_{j=1}^{\infty} [P(X_1\le a(\lambda),\ X_{j+1}\le a(\lambda)) - \lambda^2]$$

where

$$a(\lambda) = \begin{cases} a_\lambda - 0 & \text{if } F(a_\lambda)-\lambda > \lambda - F(a_\lambda - 0) \\ a_\lambda & \text{if } F(a_\lambda)-\lambda < \lambda - F(a_\lambda - 0) \end{cases}$$

and a_λ is a real such that $F(a_\lambda - 0) \le \lambda \le F(a_\lambda)$, and the event $\{X_n \le a_\lambda - 0\}$ means $\{X_n < a_\lambda\}$.

Proof: According to theorem 3.6 and Smirnov's results as above, it is sufficient to show (3.10). Writing $u_n = a_n x + b_n$, $x \in B_1(u(\cdot))$, we have

$$\left| \sum_{j=1}^{\ell_n - 1} [P(X_1 \le u_n,\ X_{j+1} \le u_n) - F^2(u_n)] - \sum_{j=1}^{\ell_n - 1} [P(X_1 \le a(\lambda),\ X_{j+1} \le a(\lambda)) - \lambda^2] \right|$$

$$\le \sum_{j=1}^{\ell_n - 1} [|P(X_1 \le u_n,\ X_{j+1} \le u_n) - P(X_1 \le a(\lambda),\ X_{j+1} \le a(\lambda))| + |F^2(u_n) - \lambda^2|]$$

$$\le 4\ell_n |F(u_n) - \lambda| \le 4\ell_n \left(\left|F(u_n) - \frac{k_n}{n}\right| + \left|\frac{k_n}{n} - \lambda\right| \right) \to 0$$

so that

$$\sum_{j=1}^{\infty} [P(X_1 \le a(\lambda),\ X_{j+1} \le a(\lambda)) - \lambda^2] = \lim_n \sum_{j=1}^{\ell_n - 1} [P(X_1 \le a(\lambda),\ X_{j+1} \le a(\lambda)) - \lambda^2]$$

$$= \lim_n \sum_{j=1}^{\ell_n - 1} [P(X_1 \le u_n,\ X_{j+1} \le u_n) - F^2(u_n)]$$

$$= \lambda \lim_n \frac{n}{k_n} \sum_{j=1}^{\ell_n - 1} E\tilde{I}_{n1} \tilde{I}_{nj+1} = \lambda\sigma .$$

Hence Theorem 3.7 holds.

§4. Example: The Normal Case

Let $\{X_n,\ n=1,2,\dots\}$ be a stationary normal sequence with

$$EX_n = 0 \quad , \quad EX_n^2 = 1 \quad , \quad EX_1 X_{n+1} = r_n \quad , \quad n=1,2,\dots .$$

In this section, we give some conditions on $\{r_n\}$ such that the limiting distributions of $\{X_{k_n}^{(n)}\}$ exist for some special rank sequences $\{k_n\}$.

Lemma 4.1 If $r_n \to 0$ and

$$(4.1)\quad \lim_n \frac{n}{k_n^{1/2}} \sum_{j=\ell_n}^{n-1} j|r_j|\exp\left(-\frac{u_n^2}{1+|r_j|}\right) = 0$$

then (2.1) holds.

Proof: The method of proving this lemma is a slight extension of an argument of Leadbetter, Lindgren and Rootzén [7]. Let $\tilde{A}_{ni} = \{x_i \le u_n\} \subset R^n$, $i=1,\ldots,n$. For any fixed integer k,ℓ, denote $F_k = \sigma\{\tilde{A}_{n1},\ldots,\tilde{A}_{nk}\}$, $F^*_{k+\ell} = \sigma\{\tilde{A}_{nk+\ell+1},\ldots,\tilde{A}_{nn}\}$. Then any $A\in\sigma\{\{X_i \le u_n\},\ i=1,\ldots,k\}$, $B\in\sigma\{\{X_i\le u_n\},\ i=k+\ell+1,\ldots,n\}$ can be represented as $A = \{X_n \in \tilde{A}\}$, $B\in\{X_n\in\tilde{B}\}$ where $\tilde{A}\in F_k$, $\tilde{B}\in F^*_{k+\ell}$. Write $f_1(x_1,\ldots,x_k;\ y_1,\ldots,y_{n-k-\ell})$ for the density of $(X_1,\ldots,X_k;\ X_{k+\ell+1},\ldots,X_n)$ and $f_0(x_1,\ldots,x_k;\ y_1,\ldots,y_{n-k-\ell}) = f_{01}(x_1,\ldots,x_k)f_{02}(y_1,\ldots,y_{n-k-\ell})$ where f_{01} and f_{02} are the densities of $(X_1,\ldots,X_k)$ and $(X_{k+\ell+1},\ldots,X_n)$ respectively. Let R_1 and R_0 be the covariance matrices of f_1 and f_0. It is easy to show that $R_h = hR_1+(1-h)R_0$ is positive definite for any $h\in[0,1]$. Writing $f_h(x_1,\ldots,x_k,y_1,\ldots,y_{n-k-\ell})$ for the density function of a zero-mean normal vector with covariance matrix R_h, we have

$$(4.2)\quad P(AB) - P(A)P(B) = \int\cdots\int_{x\in\tilde{A},y\in\tilde{B}} \left[\int_0^1 f_h'\, dh\right]dxdy$$

$$= \int_0^1 dh \sum_{\substack{1\le i\le k\\ k+\ell+1\le j\le n}} \int\cdots\int_{x\in\tilde{A},y\in\tilde{B}} \frac{\partial^2 f_h}{\partial x_i \partial y_{j-k-\ell}}\, dxdy\ ,$$

where $x = \begin{pmatrix}x_1\\ \vdots\\ x_k\end{pmatrix}$, $y = \begin{pmatrix}y_1\\ \vdots\\ y_{n-k-\ell}\end{pmatrix}$. Split the integral $\int\cdots\int_{x\in\tilde{A},y\in\tilde{B}} \frac{\partial^2 f_h}{\partial x_i \partial y_{j-k-\ell}}\, dxdy$ into four parts: for $x\in\bar{A}\cap\{x_i\le u_n\}$, $y\in\bar{B}\cap\{y_{j-k-\ell}\le u_n\}$ and $x\in\bar{A}\cap\{x_i\le u_n\}$, $y\in\bar{B}\cap\{y_{j-k-\ell}> u_n\}$.

and $x\in A\cap\{x_i>u_n\}$, $y\in B\cap\{y_{j-k-\ell}>u_n\}$, and $x\in\tilde{A}\cap\{u_i>u_n\}$, $y\in\tilde{B}\cap\{y_{j-k-\ell}>u_n\}$, where $A\in\{\tilde{A}_{n1},\dots,\tilde{A}_{ni-1},\tilde{A}_{ni+1},\dots\tilde{A}_{nk}\}$, $\tilde{B}\in\{\tilde{A}_{nk+\ell+1},\dots,\tilde{A}_{n,j-1},\tilde{A}_{n,j+1},\dots,\tilde{A}_{nn}\}$. (This can be done since $\tilde{A}$ is a disjoint union of sets of the form $\cap_{j=1}^{k}C_{nj}$, where $C_{nj}=\tilde{A}_{nj}$ or the complement of $\tilde{A}_{nj}$, and similarly for B). Writing $x^{(i)},y^{(i)}$ for the vectors x,y without the component x_i,y_i we have

$$\Big|\int_{\substack{x\in\bar{A}\cap\{x_i\le u_n\}\\ y\in\bar{B}\cap\{y_{j-k-\ell}\le u_n\}}}\!\!\cdots\int \frac{\partial^2 f_h}{\partial x_i\partial y_{j-k-\ell}}\,dxdy\Big|$$

$$\le \int_{R^{k-1}\times R^{n-k-\ell-1}}\!\!\cdots\int f_h(x_i=u_n,y_{j-k-\ell}=u_n)\,dx^{(i)}dy^{(i)}$$

$$\le \frac{1}{2\pi(1-r_{j-i}^2)^{1/2}}\exp\{-\frac{u_n^2}{1+|r_{j-i}|}\}$$

and the same inequalities hold for other three parts of the integral. Since $r_n\to 0$ we have $\sup_{n\ge 1}|r_n|<1$, and therefore

$$\Big|\int_{x\in\tilde{A},\,y\in\tilde{B}}\!\!\cdots\int\frac{\partial^2 f_h}{\partial x_i\partial y_j}\,dxdy\Big|\le C\exp(-\frac{u_n}{1+|r_{j-i}|})$$

From this and (1.2) it follows that

$$|P(AB)-P(A)P(B)|\le C\sum_{\substack{1\le i\le k\\ k+\ell<j\le n}}|r_{j-1}|\exp(-\frac{u_n}{1+|r_{j-i}|})\le C\sum_{j=\ell}^{n-1}j|r_j|\exp(-\frac{u_n}{1+|r_{j-1}|})$$

Hence (2.1) holds if (4.1) holds.

According to theorem 3.5 and 3.7 of Cheng [1], we know that (3.2) holds for any k_n satisfying (0,1), if $F(x)=\Phi(x)$ and $a_n>0$, b_n are defined by $\Phi(b_n)=\frac{k_n}{n}$, $a_n=\frac{k_n^{1/2}}{n}\cdot\frac{(1-\lambda)^{1/2}}{\phi(b_n)}$, where $\phi(x)$ is the density function of the standard normal distribution $\Phi(x)$. Using this fact, we discuss the limiting distribution of

order statistics from stationary normal sequences. Since the case $\lambda=0$ has been discussed in [1], we consider only the case $\lambda\in(0,1)$.

Lemma 4.2 If $\lambda\in(o,1)$ in (o.1) and $r_n = O(n^{-(1+\rho)})$, $\rho>3$, then (4.1) holds for some $\ell_n = o(n^{1/4})$ and any u_n which satisfies (1.1).

Proof: To show (4.1) it is sufficient to show that

$$\lim_n n^{1/2} \sum_{j=\ell_n}^{n-1} j|r_j|\exp\left(-\frac{u_n}{1+|r_j|}\right) = 0 .$$

Since $\exp\left(-\frac{u_n}{1+|r_j|}\right) \le \exp\left(-\frac{u_n^2}{2}\right)$, this will follow if

$$\lim_n n^{1/2} \sum_{j=\ell_n}^{n-1} j|r_j| \exp\left(-\frac{u_n^2}{2}\right) = 0 .$$

Since $\Phi(u_n)\to\lambda$ and the inverse function $\Phi^{-1}(x)$ of $\Phi(x)$ is continuous, we have $u_n\to a_\lambda$ where a_λ is defined by $\Phi(a_\lambda) = \lambda$. Hence

$$n^{1/2} \sum_{j=\ell_n}^{n-1} j|r_j|\exp\left(-\frac{u_n^2}{2}\right) \le \left[\exp\left(-\frac{a_\lambda^2}{2}\right) + 1\right] n^{1/2} \sum_{j=\ell_n}^{n-1} j|r_j|$$

$$\le Cn^{1/2} \sum_{j=\ell_n}^{n-1} \frac{1}{j^{\rho}} \le Cn^{1/2} \frac{\ell_n^{-\rho+1}}{\rho-1}$$

for sufficiently large n. Let $\ell_n = [n/\log n]^{1/4}$. Thus

$$n^{1/2} \sum_{j=\ell_n}^{n-1} j|r_j|\exp\left(-\frac{u_n^2}{2}\right) \le (\log n)^{\rho-1}/n^{(\rho-3)/4} \to 0$$

completing the proof of the lemma.

Lemma 4.2 If $\lambda\in(0,1)$ in $(0,1)$ and $\sum_{n=1}^{\infty}|r_n|<\infty$, then

$$(4.3)\quad \lim_{n} \sum_{j=1}^{\ell_n-1} [P(X_1\le b_n, X_{j+1}\le b_n) - (\frac{k_n}{n})^2] = \frac{1}{2\pi}\sum_{n=0}^{\infty}\int_0^{r_n} \frac{\exp(-\frac{a_\lambda^2}{1+r})}{(1-r^2)^{1/2}}\,dr$$

$$(4.4)\quad \lim_{n} \frac{1}{n^{1/2}}\sum_{j=1}^{\ell_n-1} j[P(X_1\le b_n, X_{j+1}\le b_n) - (\frac{k_n}{n})^2] = 0$$

for any $\ell_n = o(n^{1/4})$, $\ell_n\to\infty$, where a_λ is the solution of the equation $\Phi(x)=\lambda$.

Proof: It is easy to show that

$$P(X_1\le b_n, X_{j+1}\le b_n) - (\frac{k_n}{n})^2 = \frac{1}{2\pi}\int_0^{r_j}\frac{\exp(-\frac{b_n^2}{1+r})}{(1-r^2)^{1/2}}\,dr.$$

Notice that

$$|\sum_{j=1}^{\ell_n-1}\int_0^{r_j}\frac{\exp(-\frac{b_n^2}{1+r})}{(1-r^2)^{1/2}}\,dr - \sum_{j=1}^{\ell_n-1}\int_0^{r_j}\frac{\exp(-\frac{a_\lambda^2}{1+r})}{(1-r^2)^{1/2}}\,dr|$$

$$\le \sum_{j=1}^{\ell_n-1}\int_0^{|r_j|}\frac{\exp(-\frac{a_\lambda^2}{1+r})}{(1-r^2)^{1/2}}\,|\exp(-\frac{b_n^2-a_\lambda^2}{1+r}) - 1|\,dr$$

$$\le |b_n^2-a_\lambda^2|\sum_{j=1}^{\ell_n-1}\int_0^{|r_j|}\frac{\exp(-\frac{a_\lambda^2}{1+r})}{(1-r^2)^{1/2}(1+r)}\,dr$$

$$\le C|b_n^2-a_\lambda^2|\sum_{n=1}^{\infty}|r_n| \to 0 .$$

Hence to show (4.3), it is sufficient to show that the series $\sum_{n=0}^{\infty}\int_0^{r_n}\frac{\exp(-\frac{a_\lambda^2}{1+r})}{(1-r^2)^{1/2}}\,dr$ converges. This follows since

$$\left|\sum_{n=1}^{\infty} \int_0^{r_n} \frac{\exp(-\frac{a_\lambda^2}{1+r})}{(1-r^2)^{1/2}} dr\right| \le \sum_{n=1}^{\infty} |r_n| \frac{\exp(-\frac{a_\lambda^2}{1+r})}{(1-r_n^2)^{1/2}} \le C \sum_{n=1}^{\infty} |r_n| < \infty .$$

(4.3) is proved, and (4.4) can be shown in a similar way.

From lemma 4.2, 4.3 and theorem 3.6, we have

Theorem 4.4 If $r_n = 0(n^{-(1+\rho)})$, $\rho>3$, then

$$\lim_n P(X_{k_n}^{(n)} \le a_n x + b_n) = \Phi(\frac{(1-\lambda)^{1/2}}{\sigma_\lambda} x)$$

for any k_n such that $\frac{k_n}{n} \to \lambda \epsilon (o,1)$, where

$$\sigma_\lambda^2 = (1-\lambda) + \frac{1}{\pi\lambda} \sum_{n=0}^{\infty} \int_0^{r_n} \frac{\exp(-\frac{a_\lambda^2}{1+r})}{(1-r^2)^{1/2}} dr .$$

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